Brojni nizovi

Brojni niz je realna f-ja definisana nad skupom prirodnih brojeva.

Npr.

$1, 2, 3, \dots, n, n+1, \dots$ je niz prirodnih brojeva. Opšti član ovog niza je $a_n = n$, $n \in \mathbb{N}$. Niz možemo pisati i u obliku $\{n\}_{n \in \mathbb{N}}$.

$1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \frac{1}{n+1}, \dots$ je niz sa opštim članom $b_n = \frac{1}{n}$, $n \in \mathbb{N}$. Ovaj niz možemo pisati i u obliku $\left\{\frac{1}{n}\right\}_{n \in \mathbb{N}}$

$-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}, \dots$ je niz čiji je opšti član $s_n = \frac{(-1)^n}{n^2}$, $n \in \mathbb{N}$
Skraćeno niz možemo pisati kao $\left\{\frac{(-1)^n}{n^2}\right\}_{n \in \mathbb{N}}$

$\frac{1}{2}, -1, \frac{3}{2}, -2, \frac{5}{2}, -3, \dots$ je niz čiji je opšti član $t_n = \frac{(-1)^{n-1} \cdot n}{2}$
Niz možemo pisati u obliku $\left\{\frac{(-1)^{n-1}}{2} \cdot n\right\}_{n \in \mathbb{N}}$

Aritmetički niz

Aritmetički niz je niz brojeva kod kojih je razlika između dva susjedna člana stalan broj.

$a_1, a_2, a_3, a_4, \dots, a_n, a_{n+1}, \dots$

$a_2 - a_1 = d$
$a_3 - a_2 = d$
$a_4 - a_3 = d$
$\vdots$
$a_n - a_{n-1} = d$
$\vdots$

a_1
$a_2 = a_1 + d$
$a_3 = a_2 + d = a_1 + 2d$
$a_4 = a_3 + d = a_1 + 3d$
$\vdots$
$a_n = a_{n-1} + d = a_1 + (n-1)d$
$\vdots$

$s + t = n + 1$
$a_s + a_t = a_1 + (s-1)d + a_1 + (t-1)d =$
$= 2a_1 + (s+t-2)d = 2a_1 + (n-1)d$
$= a_1 + a_n$

$$S_n = a_1 + a_2 + \dots + a_n$$
$$+ \quad S_n = a_n + a_{n-1} + \dots + a_1$$
$$2S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + \dots + (a_n + a_1)$$
$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(2a_1 + (n-1)d)$$
suma prvih n članova

1. Izračunati sumu prvih 20 članova niza $2, 5, 8, 11, 14, \dots$

Rj. Ovo je aritmetički niz, $d = 3$

$a_{20} = a_{19} + 3 = a_1 + 19 \cdot 3 = 2 + 57 = 59$

$S_n = \frac{n}{2}(a_1 + a_n) = \frac{20}{2}(2 + 59) = 10 \cdot 61 = 610$ suma prvih dvadeset članova

Geometrijski niz

Geometrijski niz je niz brojeva kod kojeg je količnik dva susjedna člana stalan broj.

$$b_1,\ b_2,\ b_3,\ b_4,\ \dots\ b_{n-1},\ b_n,\ \dots$$

$b_2 : b_1 = q$

$b_3 : b_2 = q$

$b_4 : b_3 = q$

$\vdots$

$b_n : b_{n-1} = q$

$\vdots$

b_1

$b_2 = b_1 q$

$b_3 = b_2 q = b_1 q^2$

$b_4 = b_3 q = b_1 q^3$

$\vdots$

$b_n = b_{n-1} q = b_1 q^{n-1}$

$\vdots$

$$S_n = b_1 + b_2 + b_3 + \dots + b_n$$

$$S_n = b_1 + b_1 q + b_1 q^2 + \dots + b_1 q^{n-1}$$

$$S_n = b_1(1 + q + q^2 + \dots + q^{n-1}) \ /(1-q)$$

$$(1-q)S_n = b_1(1-q)(1+q+q^2+\dots+q^{n-1})$$

$$(1-q)S_n = b_1(1-q^n) \quad /:(1-q)$$

$$S_n = b_1 \frac{1-q^n}{1-q}$$ suma prvih n članova

(2.) Izračunati sumu prvih 50 članova niza $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$

Rj. Ovo je geometrijski niz. $b_1 = \frac{1}{3}$, $q = \frac{1}{3}$, $S_n = b_1 \frac{1-q^n}{1-q}$.

$$S_{50} = \frac{1}{3} \cdot \frac{1-(\frac{1}{3})^{50}}{1-\frac{1}{3}} = \frac{1}{3}\cdot\frac{3}{2}\cdot\left(1-\frac{1}{3^{50}}\right) = \frac{1}{2}\left(1-\frac{1}{3^{50}}\right) = \frac{1}{2} - \frac{1}{2\cdot 3^{50}} \approx \frac{1}{2}$$

Monotoni nizovi

Ako je $x_n < x_{n+1}$ tada niz $\{x_n\}_{n\in\mathbb{N}}$ raste

$x_n \le x_{n+1} \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ ne opada

$x_n > x_{n+1} \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ opada

$x_n \ge x_{n+1} \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ ne raste

ove nizove jednim imenom zovemo monotoni nizovi

$$a_{n+1} - a_n = \dots \begin{cases} <0, & \text{niz opada} \\ >0, & \text{niz raste} \end{cases} \qquad \frac{a_{n+1}}{a_n} = \dots \begin{cases} >1, & \text{rastući niz} \\ <1, & \text{opadajući niz} \end{cases}$$

(3.) Ispitati monotonost niza $\{a_n\}_{n\in\mathbb{N}}$ gdje je $a_n = \frac{n-1}{2n+1}$.

Rj. $$a_{n+1} - a_n = \frac{n+1-1}{2(n+1)+1} - \frac{n-1}{2n+1} = \frac{n}{2n+3} - \frac{n-1}{2n+1} = \frac{2n^2+n-(2n^2-2n+3n-3)}{(2n+3)(2n+1)}$$

$$= \frac{3}{(2n+3)(2n+1)} > 0, \ \forall n \quad \Rightarrow \{a_n\} \text{ je rastući niz}$$

Granična vrijednost niza

Broj A nazivamo granična vrijednost niza ili limesom niza realnih brojeva $x_1, x_2, \dots, x_n, \dots$, što simbolički pišemo

$$\lim_{n\to\infty} x_n = A$$

ako za svaki $\varepsilon > 0$ postoji broj N (koji zavisi od ε) tako da

$$|x_n - A| < \varepsilon \quad \text{za svaki } n > N.$$

1. Dat je niz $1, \frac{1}{4}, \frac{1}{9}, \dots, \frac{1}{n^2}, \dots$ Izračunati za koju vrijednost n će biti zadovoljena nejednakost $\frac{1}{n^2} < \varepsilon$ ako je $\varepsilon = 0{,}001$.

Rj.

$$\frac{1}{n^2} < 0{,}001 \qquad \frac{1}{n^2} < 10^{-3} \quad / \cdot n^2$$

$$10^{-3} n^2 > 1 \quad / \cdot 10^3 \qquad n^2 > 10^3 \qquad n > 10\sqrt{10} \approx 31{,}62$$

Za sve $n > 31$ će biti zadovoljena nejednakost $\frac{1}{n^2} < \varepsilon$.

2. Pokazati da je $\lim_{n\to\infty} \frac{2n+1}{n+1} = 2$.

Rj. Iz definicije $\forall \varepsilon > 0 \ \exists N$ (koji zavisi od ε) tako da

$$\left| \frac{2n+1}{n+1} - 2 \right| < \varepsilon \quad \text{za svaki } n > N.$$

$$\left| \frac{2n+1}{n+1} - 2 \right| = \left| \frac{2n+1-2n-2}{n+1} \right| = \left| \frac{-1}{n+1} \right| = \frac{1}{n+1} < \varepsilon$$

$$(n+1)\varepsilon > 1 \quad / : \varepsilon \ (\varepsilon > 0) \qquad n+1 > \frac{1}{\varepsilon} \qquad n > \frac{1}{\varepsilon} - 1$$

Prema tome za svaki pozitivan broj ε postoji takav broj N ($N = \frac{1}{\varepsilon} - 1$) takav da za $n > N$ vrijedi $\left| \frac{2n+1}{n+1} - 2 \right| < \varepsilon$.
Prema tome $\lim_{n\to\infty} \frac{2n+1}{n+1} = 2$.

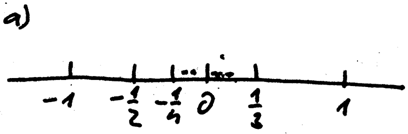
3. Odredite limese nizova

a) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, \frac{(-1)^{n-1}}{n}, \dots$;

b) $\frac{2}{1}, \frac{4}{3}, \frac{6}{5}, \dots, \frac{2n}{2n-1}, \dots$;

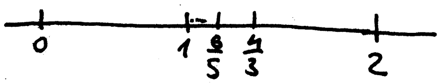
c) $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots$;

Rj. a)



0

b)



1

c) $\sqrt{2} \approx 1{,}41$

$\sqrt{2\sqrt{2}} = \sqrt[4]{8} \approx 1{,}68$

$\sqrt{2\sqrt{2\sqrt{2}}} = \sqrt{2\sqrt[4]{8}} = \sqrt[8]{2^7} \approx 1{,}83$

(#) Dat je niz $v_1 = \frac{\cos\frac{\pi}{2}}{1}$, $v_2 = \frac{\cos\pi}{2}$, $v_3 = \frac{\cos\frac{3\pi}{2}}{3}$, ..., $v_n = \frac{\cos\frac{n\pi}{2}}{n}$. Nađi $\lim_{n\to\infty} v_n$. Koliko mora biti n da bi apsolutna vrijednost razlike između v_n i $\lim_{n\to\infty} v_n$ bila ne veća od 0,0001?

R.: $\cos\frac{\pi}{2}=0$, $\cos\pi=-1$, $\cos\frac{3\pi}{2}=0$, $\cos 2\pi = 1$, ...

Kolko iznosi $\cos\frac{n\pi}{2}$.

Za n neparno tj. za n oblika $n=2k+1$, $k\in\mathbb{N}$

$$\cos\frac{(2k+1)\pi}{2} = 0$$

Za n parno tj. za n oblika $n=2k$, $k\in\mathbb{N}$

$$\cos\frac{2k\pi}{2} = (-1)^{k+1}$$

Napišimo sad neke članove niza (v_n)

$$0,\ \frac{-1}{2},\ 0,\ \frac{1}{4},\ 0,\ -\frac{1}{6},\ 0,\ \frac{1}{8},\ 0,\ \frac{-1}{10},\ 0, \dots \qquad \dots(*)$$

Prema definiciji $\lim_{n\to\infty} v_n = A$ ako i samo ako $\forall\varepsilon>0$ $\exists$ privodan broj N (N zavisi samo od ε) ~~takav~~ da

$$|v_n - A| < \varepsilon \quad \text{za svaki } n>N.$$

$$\left|\frac{\cos\frac{n\pi}{2}}{n} - A\right| < \varepsilon \iff -\varepsilon < \frac{\cos\frac{n\pi}{2}}{n} - A < \varepsilon$$

$$A-\varepsilon < \frac{\cos\frac{n\pi}{2}}{n} < A+\varepsilon$$

Iz ove definicije ili iz $(*)$ možemo zaključiti da je $A=0$ tj. $\lim_{n\to\infty} v_n = 0$. Uzmimo da je $\varepsilon = 0{,}0001 = 10^{-4}$.

$$\left|\frac{\cos\frac{n\pi}{2}}{n} - 0\right| < 10^{-4}$$

Kako je $|\cos\frac{n\pi}{2}|\le 1$ $\forall n$ imamo

$$\frac{1}{n} < 10^{-4} \Rightarrow n > \frac{1}{10^{-4}} = 10^4.$$

Za svako $n>10^4$ apsolutn. vrijednost razlike između v_n i $\lim_{n\to\infty} v_n$ je manja od 0,0001.

(#) Naći vrijednost sljedećih f-ja

a) $f(x) = 2x - 3 - \frac{1}{x}$ kada $x \to 1$;

b) $f(x) = \frac{x^3 - 3x^2 + 2x - 5}{x^2 + 2}$ kada $x \to -1$

c) $y = x \sin \frac{1}{x}$ kada $x \to 0$.

Rj:

a) $\lim_{x \to 1} (2x - 3 - \frac{1}{x}) = 2 \cdot 1 - 3 - 1 = -2$;

b) $\lim_{x \to -1} \frac{x^3 - 3x^2 + 2x - 5}{x^2 + 2} = \frac{(-1)^3 - 3(-1)^2 + 2 \cdot (-1) - 5}{(-1)^2 + 2} =$

$= \frac{-1 - 3 - 2 - 5}{1 + 2} = \frac{-11}{3}$

c) Primjetimo da $\lim_{x \to 0} \frac{1}{x} = \infty$

Pa je $\lim_{x \to 0} \sin \frac{1}{x}$ $\underbrace{\text{neodređen izraz}}_{}$ (ali uvijek je između -1 i 1).

Međutim kako je $|\sin \frac{1}{x}| \leq 1$ za svako x i nula pomnožena sa bilo kojim konačnim brojem je nula to

$\lim_{x \to 0} x \cdot \sin \frac{1}{x} = 0$.

Napomena: $0 \cdot \infty$ je neodređen izraz

$\sqrt{2},\ \sqrt[4]{2^3},\ \sqrt[8]{2^7},\ \ldots,\ \sqrt[2^n]{2^{2^n-1}},\quad \lim\limits_{n\to\infty} 2^{\frac{2^n-1}{2^n}} = 1$

Operacije sa limesima

a) $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

b) $\lim\limits_{n\to\infty}(a_n \cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

c) $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$

d) $\lim\limits_{n\to\infty}\sqrt[k]{a_n} = \sqrt[k]{\lim\limits_{n\to\infty} a_n}$

e) $\lim\limits_{n\to\infty} b^{a_n} = b^{\lim\limits_{n\to\infty} a_n},\ b>0$

f) $\lim\limits_{n\to\infty}\log_b a_n = \log_b \lim\limits_{n\to\infty} a_n,\ b>1$

1. Izračunajte limese

a) $\lim\limits_{n\to\infty}\frac{1}{n}$ Rj. $\lim\limits_{n\to\infty}\frac{1}{n} = 0$

b) $\lim\limits_{n\to\infty} 7$ Rj. $\lim\limits_{n\to\infty} 7 = 7$

c) $\lim\limits_{n\to\infty} n^2$ Rj. $\lim\limits_{n\to\infty} n^2 = \infty$

d) $\lim\limits_{n\to\infty}\dfrac{n}{n+1}$

Rj. $\lim\limits_{n\to\infty}\dfrac{n}{n+1}\left(=\frac{\infty}{\infty}\right)\begin{matrix}/:n\\/:n\end{matrix} = \lim\limits_{n\to\infty}\dfrac{1}{1+\underbrace{\frac{1}{n}}_{\to 0}} = 1$

e) $\lim\limits_{n\to\infty}\dfrac{n^2+n-3}{n^3+n^2+1}\begin{matrix}/:n^3\\/:n^3\end{matrix}$ Rj. 0

Neodređeni izrazi su $\frac{0}{0},\ \infty-\infty,\ 0\cdot\infty,\ \frac{\infty}{\infty},\ \frac{\infty}{0}$

Određeni izrazi su $\infty\cdot\infty=\infty,\ \infty+\infty=\infty,\ \frac{0}{\infty}=0$

2. Izračunati limese:

a) $\lim\limits_{n\to\infty}\dfrac{n^3+3n+9}{2n^2+3n-1}$ Rj. $\lim\limits_{n\to\infty}\dfrac{n^3+3n+9}{2n^2+3n-1}\begin{matrix}/:n^3\\/:n^3\end{matrix} = \lim\limits_{n\to\infty}\dfrac{1+\overbrace{\frac{3}{n^2}}^{\to 0}+\overbrace{\frac{9}{n^3}}^{\to 0}}{\underbrace{\frac{2}{n}}_{\to 0}+\underbrace{\frac{3}{n^2}}_{\to 0}-\underbrace{\frac{1}{n^3}}_{\to 0}} = \dfrac{1}{0} = \infty$

b) $\lim\limits_{n\to\infty}\dfrac{n^2+2n+3}{2n^2+n-4}$ Rj. $\lim\limits_{n\to\infty}\dfrac{n^2+2n+3}{2n^2+n-4}\begin{matrix}/:n^2\\/:n^2\end{matrix} = \lim\limits_{n\to\infty}\dfrac{1+\frac{2}{n}+\frac{3}{n^2}}{2+\frac{1}{n}-\frac{4}{n^2}} = \dfrac{1}{2}$

c) $\lim\limits_{n\to\infty}\dfrac{3n^3+n-1}{2n^4+1}$ Rj. $\lim\limits_{n\to\infty}\dfrac{3n^3+n-1}{2n^4+1}\begin{matrix}/:n^4\\/:n^4\end{matrix} = \lim\limits_{n\to\infty}\dfrac{\frac{3}{n}+\frac{1}{n^3}-\frac{1}{n^4}}{2+\frac{1}{n^4}} = \dfrac{0}{2} = 0$

d) $\lim\limits_{n\to\infty}\dfrac{(n+1)(n+2)(n+3)}{n^3}$ Rj. $\lim\limits_{n\to\infty}\dfrac{(n+1)(n+2)(n+3)}{n^3}\begin{matrix}/:n^3\\/:n^3\end{matrix} = \lim\limits_{n\to\infty}\dfrac{(1+\frac{1}{n})(1+\frac{2}{n})(1+\frac{3}{n})}{1} = \dfrac{1}{1} = 1$

e) $\lim\limits_{n\to\infty}\dfrac{n+(-1)^n}{3n-(-1)^n}$ Rj. $\lim\limits_{n\to\infty}\dfrac{n+(-1)^n}{3n-(-1)^n}\begin{matrix}/:n\\/:n\end{matrix} = \lim\limits_{n\to\infty}\dfrac{1+\overbrace{\frac{(-1)^n}{n}}^{\to 0}}{3-\underbrace{\left(\frac{(-1)^n}{n}\right)}_{\to 0}} = \dfrac{1}{3}$

3. Izračunati limese:

a) $\lim\limits_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\ldots+\frac{n-1}{n^2}\right)$ b) $\lim\limits_{n\to\infty}\left(\frac{1+3+5+\ldots+(2n-1)}{n+1}-\frac{2n+1}{2}\right)$

c) $\lim\limits_{n\to\infty}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}\right)$ d) $\lim\limits_{n\to\infty}\left(1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\ldots+\frac{(-1)^{n-1}}{3^{n-1}}\right)$

Rj. a) rj. $\frac{1}{2}$ c) rj. 1

b) $\lim\limits_{n\to\infty}\left(\frac{1+3+5+\ldots+(2n-1)}{n+1}-\frac{2n+1}{2}\right)=\lim\limits_{n\to\infty}\left(\frac{\frac{n}{2}(1+2n-1)}{n+1}-\frac{2n+1}{2}\right)=\lim\limits_{n\to\infty}\left(\frac{2n^2}{2n+2}-\frac{2n+1}{2}\right)$

$=\lim\limits_{n\to\infty}\frac{2n^2-\overbrace{(2n+1)(n+1)}^{2n^2+3n+1}}{2(n+1)}=\lim\limits_{n\to\infty}\frac{\cancel{2n^2}-\cancel{2n^2}-3n-1}{2n+2}\begin{matrix}/:n\\/:n\end{matrix}=\lim\limits_{n\to\infty}\frac{-3-\frac{1}{n}}{2+\frac{2}{n}}=-\frac{3}{2}$

d) imamo niz $1,-\frac{1}{3},\frac{1}{9},-\frac{1}{27},\ldots$ količnik dva susjedna člana je $-\frac{1}{3}$

imamo geometrijski niz, $|q|<1$, $S_n=a_1\frac{1-q^n}{1-q}$

$\lim\limits_{n\to\infty}\left(1-\frac{1}{3}+\frac{1}{9}-\ldots+\frac{(-1)^{n-1}}{3^{n-1}}\right)=\lim\limits_{n\to\infty}\left(1\cdot\frac{1-\overbrace{\left(-\frac{1}{3}\right)^n}^{\to 0}}{1-\left(-\frac{1}{3}\right)}\right)=\frac{1}{1+\frac{1}{3}}=\frac{1}{\frac{4}{3}}=\frac{3}{4}$

4. Izračunati limese:

a) $\lim\limits_{n\to\infty}\frac{2^{n+1}+3^{n+1}}{2^n+3^n}$ b) $\lim\limits_{n\to\infty}(\sqrt{n+1}-\sqrt{n})$ c) $\lim\limits_{n\to\infty}\frac{n\sin n!}{n^2+1}$

d) $\lim\limits_{x\to\infty}\frac{(2x-3)(3x+5)(4x-6)}{3x^3+x-1}$ e) $\lim\limits_{x\to\infty}\frac{1000x}{x^2-1}$ f) $\lim\limits_{x\to\infty}\frac{2x^2-x^3-4}{\sqrt{x^4+1}}$

g) $\lim\limits_{x\to\infty}\frac{2x+3}{x+\sqrt[3]{x}}$ h) $\lim\limits_{x\to\infty}\frac{x^2}{10+x\sqrt{x}}$ i) $\lim\limits_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$

Rj. a) $\lim\limits_{n\to\infty}\frac{2^{n+1}+3^{n+1}}{2^n+3^n}\begin{matrix}/:3^n\\/:3^n\end{matrix}=\lim\limits_{n\to\infty}\frac{\overbrace{\frac{2^{n+1}}{3^n}}^{0}+3}{\underbrace{\left(\frac{2}{3}\right)^n}_{0}+1}=3$

b) $\lim\limits_{n\to\infty}(\sqrt{n+1}-\sqrt{n})\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\lim\limits_{n\to\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\infty}=0$

i) $\lim\limits_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}=\lim\limits_{x\to+\infty}\left(\frac{x}{x+\sqrt{x+\sqrt{x}}}\right)^{\frac{1}{2}}=\lim\limits_{x\to+\infty}\left(\frac{1}{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}\right)^{\frac{1}{2}}=1$

c) rj. 0 d) rj. 8 e) rj. 0 f) rj. 2 g) rj. 2 h) ∞

Granična vrijednost f-je

Kažemo da f-ja $f(x) \to A$ kada $x \to p$ (A i p su brojevi) ili da je $\lim\limits_{x\to a} f(x) = A$ ako za svaki $\varepsilon > 0$ postoji takav $\delta > 0$ (δ zavisi od ε) da je $|f(x) - A| < \varepsilon$ za $0 < |x - p| < \delta$.

1. Izračunati limese:

a) $$\lim_{x\to 2} \frac{x^2-4}{x^2-3x+2} \left(=\frac{0}{0}\right) = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x-1)} = \lim_{x\to 2} \frac{x+2}{x-1} = 4$$

b) $$\lim_{x\to -1} \frac{x^3+1}{x^2+1} = \frac{0}{2} = 0$$

c) $$\lim_{x\to 5} \frac{x^2-5x+10}{x^2-25} = \frac{25-25+10}{0} = \infty$$

d) $$\lim_{x\to -1} \frac{x^2-1}{x^2+3x+2} \left(=\frac{0}{0}\right) = \lim_{x\to -1} \frac{(x-1)(x+1)}{(x+1)(x+2)} = \lim_{x\to -1} \frac{x-1}{x+2} = \frac{-2}{1} = -2$$

e) $$\lim_{x\to 2} \frac{x^2-2x}{x^2-4x+4} \left(=\frac{0}{0}\right) = \lim_{x\to 2} \frac{x(x-2)}{(x-2)^2} = \lim_{x\to 2} \frac{x}{x-2} = \frac{2}{0} = +\infty$$

f)$^\vee$ $$\lim_{x\to 1} \frac{x^3-3x+2}{x^4-4x+3}$$ Rj. $\frac{1}{2}$

g) $$\lim_{x\to a} \frac{x^2-(a+1)x+a}{x^3-a^3} \left(=\frac{0}{0}\right) = \lim_{x\to a} \frac{(x-a)(x-1)}{(x-a)(x^2+ax+a^2)} = \frac{a-1}{a^2+a^2+a^2} = \frac{a-1}{3a^2}$$

$$(x^2 \overbrace{-(a+1)x}^{-ax-x} + a) : (x-a) = x-1$$
$$\underline{-\ x^2 - ax}$$
$$-x+a$$
$$\underline{-x+a}$$
$$= \quad =$$

1
1 1
1 2 1
1 3 3 1

h) $$\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \left(=\frac{0}{0}\right) = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} = \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2$$

i)$^\vee$ $$\lim_{x\to 1}\left(\frac{1}{1-x} - \frac{3}{1-x^3}\right)$$ Rj. -1.

2. Izračunati limese

a) $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}\left(=\frac{0}{0}\right)=\left|\begin{array}{l}\text{uvodimo smjenu}\\ 1+x=y^6\\ x\to 0\Rightarrow y\to 1\end{array}\right|=\lim\limits_{y\to1}\frac{y^3-1}{y^2-1}=\lim\limits_{y\to1}\frac{(y-1)(y^2+y+1)}{(y-1)(y+1)}=\frac{3}{2}$

b) $\lim\limits_{x\to1}\frac{\sqrt{x}-1}{x-1}\left(=\frac{0}{0}\right)=\left|\begin{array}{l}x=t^2\\ x\to1\Rightarrow t\to1\end{array}\right|=\lim\limits_{t\to1}\frac{t-1}{t^2-1}=\lim\limits_{t\to1}\frac{t-1}{(t-1)(t+1)}=\frac{1}{2}$

c) $\lim\limits_{x\to64}\frac{\sqrt{x}-8}{\sqrt[3]{x}-4}$ Rj. 3

d) $\lim\limits_{x\to1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}\left(=\frac{0}{0}\right)=\left|\begin{array}{l}x=t^{12}\\ x\to1\Rightarrow t\to1\end{array}\right|=\lim\limits_{t\to1}\frac{t^4-1}{t^3-1}=\lim\limits_{t\to1}\frac{\overbrace{\cancel{(t-1)}(t+1)}^{(t^2-1)}(t^2+1)}{\cancel{(t-1)}(t^2+t+1)}=\frac{4}{3}$

e) $\lim\limits_{x\to1}\frac{\sqrt[3]{x^2}-2\sqrt[3]{x}+1}{(x-1)^2}$ Rj. $\frac{1}{9}$

3. Izračunati limese

a) $\lim\limits_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}\left(=\frac{0}{0}\right)=\lim\limits_{x\to a}\frac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{(x-a)(\sqrt{x}+\sqrt{a})}=\lim\limits_{x\to a}\frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})}=\frac{1}{2\sqrt{a}}$ $(a>0)$

b) $\lim\limits_{x\to7}\frac{2-\sqrt{x-3}}{x^2-49}\left(=\frac{0}{0}\right)=\lim\limits_{x\to7}\frac{(2-\sqrt{x-3})(2+\sqrt{x-3})}{\underbrace{(x^2-49)}_{(x-7)(x+7)}(2+\sqrt{x-3})}=\lim\limits_{x\to7}\frac{\cancel{7-x}}{(-1)\cancel{(7-x)}\underbrace{(x+7)}_{14}\underbrace{(2+\sqrt{x-3})}_{4}}=-\frac{1}{56}$

c) $\lim\limits_{x\to8}\frac{x-8}{\sqrt[3]{x}-2}$ Rj. 12

d) $\lim\limits_{x\to1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}\left(=\frac{0}{0}\right)=\lim\limits_{x\to1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}{(\sqrt[3]{x}-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)(\sqrt{x}+1)}=\lim\limits_{x\to1}\frac{\cancel{(x-1)}(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}{\cancel{(x-1)}(\sqrt{x}+1)}=\frac{3}{2}$

e) $\lim\limits_{x\to4}\frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}\left(=\frac{0}{0}\right)=\lim\limits_{x\to4}\frac{(3-\sqrt{5+x})(3+\sqrt{5+x})(1+\sqrt{5-x})}{(1-\sqrt{5-x})(1+\sqrt{5-x})(3+\sqrt{5+x})}=\lim\limits_{x\to4}\frac{(4-x)(1+\sqrt{5-x})}{\underbrace{(-4+x)}_{(-1)(4-x)}(3+\sqrt{5+x})}=\frac{2}{-6}=-\frac{1}{3}$

f) $\lim\limits_{x\to0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$ Rj. 1

g) $\lim\limits_{h\to0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\left(=\frac{0}{0}\right)=\lim\limits_{h\to0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h\cdot(\sqrt{x+h}+\sqrt{x})}=\lim\limits_{h\to0}\frac{x+h-x}{h\cdot(\sqrt{x+h}+\sqrt{x})}=\frac{1}{2\sqrt{x}}$

h) $\lim\limits_{h\to0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$ $(x\neq0)$, Rj. $\frac{1}{3\sqrt[3]{x^2}}$

i) $\lim\limits_{x\to3}\frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}$ Rj. $-\frac{1}{3}$

4. Izračunati limese

a) $\lim\limits_{x\to+\infty}(\sqrt{x+a}-\sqrt{x})\ (=\infty-\infty)=\lim\limits_{x\to+\infty}\dfrac{(\sqrt{x+a}-\sqrt{x})(\sqrt{x+a}+\sqrt{x})}{(\sqrt{x+a}+\sqrt{x})}=\lim\limits_{x\to+\infty}\dfrac{x+a-x}{(\sqrt{x+a}+\sqrt{x})}=\dfrac{a}{+\infty}=0$

b) $\lim\limits_{x\to+\infty}[\sqrt{x(x+a)}-x]\ (=\infty-\infty)=\lim\limits_{x\to+\infty}\dfrac{[\sqrt{x(x+a)}-x][\sqrt{x(x+a)}+x]}{\sqrt{x(x+a)}+x}=\lim\limits_{x\to+\infty}\dfrac{x^2+ax-x^2}{\sqrt{x(x+a)}+x}=$

$=\lim\limits_{x\to+\infty}\dfrac{ax \quad /:x}{\sqrt{x(x+a)}+x \quad /:x}=\lim\limits_{x\to+\infty}\dfrac{a}{\sqrt{1\cdot(1+\underbrace{\frac{a}{x}}_{\to 0})}+1}=\dfrac{a}{2}$

c)$^{\vee}$ $\lim\limits_{x\to+\infty}(\sqrt{x^2-5x+6}-x)$ Rj. $-\dfrac{5}{2}$

d) $\lim\limits_{x\to+\infty}x(\sqrt{x^2+1}-x)\ (=\infty(\infty-\infty))=\lim\limits_{x\to+\infty}\dfrac{x(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{(\sqrt{x^2+1}+x)}=\lim\limits_{x\to+\infty}\dfrac{x(x^2+1-x^2)}{(\sqrt{x^2+1}+x)}$

$=\lim\limits_{x\to\infty}\dfrac{x \quad /:x}{\sqrt{x^2+1}+x \quad /:x}=\lim\limits_{x\to\infty}\dfrac{1}{\sqrt{1+\underbrace{\frac{1}{x^2}}_{\to 0}}+1}=\dfrac{1}{2}$

e)$^{\vee}$ $\lim\limits_{x\to\infty}(x+\sqrt[3]{1-x^3})$ Rj. 0

Navedimo nekoliko važnih graničnih vrijednosti:

$\lim\limits_{x\to0}\dfrac{\sin x}{x}=1$ $\qquad$ $\lim\limits_{x\to0}(1+x)^{\frac{1}{x}}=e$

$\lim\limits_{x\to\infty}\left(1+\dfrac{k}{x}\right)=e^k$ $\qquad$ $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$

$\lim\limits_{n\to\infty}\sqrt[n]{a}=1$ $\qquad$ $\lim\limits_{n\to\infty}\dfrac{a^n}{n}=\infty$ $\qquad$ $\lim\limits_{n\to\infty}\dfrac{n^k}{a^n}=0$

5. Izračunati limese

a) $\lim\limits_{x\to0}\dfrac{\sin 5x}{x}=\lim\limits_{x\to0}\left(\dfrac{\sin 5x}{5x}\cdot5\right)=1\cdot5=5$

b) $\lim\limits_{x\to2}\dfrac{\sin x}{x}=\dfrac{1}{2}\sin 2$

c) $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=\left|\begin{array}{l}\text{kako je}\\ -1\le\sin\alpha\le1\\ \text{za } \forall\alpha\end{array}\right|=0$

d)$^{\vee}$ $\lim\limits_{x\to0}\dfrac{\sin 3x}{x}$ Rj. 3

e) $\lim\limits_{x\to0}\dfrac{\sin 5x}{\sin 2x}=\lim\limits_{x\to0}\dfrac{\frac{\sin 5x}{5x}\cdot5}{\frac{\sin 2x}{2x}\cdot2}=\dfrac{5}{2}$

e) $\lim\limits_{x\to\pi}\frac{\sin mx}{\sin nx} = \left| \begin{matrix} x=\pi+t \\ x\to\pi \Rightarrow t\to 0\end{matrix}\right| = \lim\limits_{t\to0}\frac{\sin(m\pi+mt)}{\sin(n\pi+nt)} = \lim\limits_{t\to0}\frac{\overbrace{\sin m\pi}^{=0}\cos mt+\sin mt\cos m\pi}{\underbrace{\sin n\pi}_{=0}\cos nt+\sin nt\cos n\pi}$

$= \lim\limits_{t\to0}\frac{(-1)^m \sin mt}{(-1)^n \sin nt} = (-1)^{m-n}\lim\limits_{t\to0}\frac{\frac{\sin mt}{mt}\cdot mt}{\frac{\sin nt}{nt}\cdot nt} = (-1)^{m-n}\frac{m}{n}$

f) $\lim\limits_{x\to0}\frac{1-\cos x}{x^2} = \lim\limits_{x\to0}\frac{2\sin^2\frac{x}{2}}{x^2} = \lim\limits_{x\to0}\frac{2\left(\sin\frac{x}{2}\right)^2}{4\cdot\left(\frac{x}{2}\right)^2} = \lim\limits_{x\to0}\frac{1}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}$

$\left.\begin{matrix} 1=\sin^2 x+\cos^2 x & \quad 1=\sin^2\frac{x}{2}+\cos^2\frac{x}{2} \\ \cos 2x=\cos^2 x-\sin^2 x & \quad \cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\end{matrix}\right\}- \Rightarrow 1-\cos x = 2\sin^2\frac{x}{2}$

g) $\lim\limits_{x\to0}\frac{\operatorname{tg} x}{x} = \lim\limits_{x\to0}\frac{\frac{\sin x}{\cos x}}{x} = \lim\limits_{x\to0}\frac{\sin x}{x}\cdot\frac{1}{\cos x} = 1$

h) $\lim\limits_{x\to1}\frac{\sin\pi x}{\sin 3\pi x}$ Rj. $\frac{1}{3}$ i) $\lim\limits_{n\to\infty}\left(n\sin\frac{\pi}{n}\right)$ Rj. π

j) $\lim\limits_{x\to0}\frac{\sin 5x-\sin 3x}{\sin x} = \lim\limits_{x\to0}\frac{5\cdot\frac{\sin 5x}{5x}-\frac{\sin 3x}{3x}\cdot 3}{\frac{\sin x}{x}} = 5-3=2$

k) $\lim\limits_{x\to a}\frac{\sin x-\sin a}{x-a} = \lim\limits_{x\to a}\frac{2\sin\frac{x-a}{2}\cos\frac{x+a}{2}}{x-a} = \lim\limits_{x\to a}\frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}}\cdot\cos\frac{x+a}{2} = \cos a$

$\left.\begin{matrix} \sin x = \sin\left(\frac{x-a}{2}+\frac{x+a}{2}\right) = \sin\frac{x-a}{2}\cos\frac{x+a}{2}+\sin\frac{x+a}{2}\cos\frac{x-a}{2} \\ -\sin a = \sin(-a) = \sin\left(\frac{x-a}{2}-\frac{x+a}{2}\right) = \sin\frac{x-a}{2}\cos\frac{x+a}{2}-\sin\frac{x+a}{2}\cos\frac{x-a}{2}\end{matrix}\right\}+$

$\sin x-\sin a = 2\sin\frac{x-a}{2}\cos\frac{x+a}{2}$

6) Izračunati limese

a) $\lim\limits_{x\to\infty}\left(\frac{x-1}{x+1}\right)^x = \lim\limits_{x\to\infty}\left(\frac{\frac{x-1}{x}}{\frac{x+1}{x}}\right)^x = \lim\limits_{x\to\infty}\frac{\left(1-\frac{1}{x}\right)^x}{\left(1+\frac{1}{x}\right)^x} = \frac{\lim\limits_{x\to\infty}\left(1+\frac{-1}{x}\right)^x}{e} = \frac{e^{-1}}{e} = e^{-2}$

b) $\lim\limits_{x\to0}\left(\frac{2+x}{3-x}\right)^x = \left(\frac{2}{3}\right)^0 = 1$

c) $\lim\limits_{x\to\infty}\left(\frac{x+1}{2x+1}\right)^{x^2} = \lim\limits_{x\to\infty}\left(\frac{1+\frac{1}{x}}{2+\frac{1}{x}}\right)^{x^2} = \left(\frac{1}{2}\right)^{+\infty} = 0$

d) $\lim\limits_{x\to1}\left(\frac{x-1}{x^2-1}\right)^{x+1}$ Rj. $\frac{1}{4}$ e) $\lim\limits_{x\to\infty}\left(\frac{1}{x^2}\right)^{\frac{2x}{x+1}}$ Rj. 0

Izračunati limes $\lim_{n\to\infty}\left(\frac{1+2+3+\dots+(n-1)}{n+1}-\frac{n}{2}\right)$

Rj.

$$1+2+3+\dots+(n-1)=\frac{n-1}{2}(1+(n-1)) \leftarrow \text{suma aritmetičkog niza}$$

$$=\frac{n-1}{2}\cdot n=\frac{n(n-1)}{2}$$

$$\lim_{n\to\infty}\left(\frac{1+2+3+\dots+(n-1)}{n+1}-\frac{n}{2}\right)=\lim_{n\to\infty}\left(\frac{\frac{n(n-1)}{2}}{n+1}-\frac{n}{2}\right)=$$

$$=\lim_{n\to\infty}\left(\frac{n(n-1)}{2(n+1)}-\frac{n}{2}\right)=\lim_{n\to\infty}\frac{n(n-1)-n(n+1)}{2(n+1)}=$$

$$=\lim_{n\to\infty}\frac{n^2-n-n^2-n}{2n+2}=\lim_{n\to\infty}\frac{-2n}{2(n+1)}=\lim_{n\to\infty}\frac{-n\ :n}{n+1\ :n}\left(=\frac{\infty}{\infty}\right)$$

$$=\lim_{n\to\infty}\frac{-1}{1+\underbrace{\frac{1}{n}}_{0}}=-1$$

Izračunati $\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{1-x}$.

Rj. $a^3-b^3=(a-b)(a^2+ab+b^2)$

$$(\sqrt[3]{x}-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)=(\sqrt[3]{x})^3-1^3=x-1$$

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{1-x}\left(\frac{0}{0}\right)=-\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}\cdot\frac{(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}{(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}=-\lim_{x\to 1}\frac{\cancel{(x-1)}}{\cancel{(x-1)}(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}$$

$$=-\lim_{x\to 1}\frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}=\frac{-1}{\sqrt[3]{1}+\sqrt[3]{1}+1}=-\frac{1}{3}$$

(#) Izračunati limese

a) $\lim\limits_{n\to\infty}\left(1+\frac{a}{n}\right)^{n}$; b) $\lim\limits_{x\to 0}\sqrt[x]{1-2x}$;

c) $\lim\limits_{t\to\infty}\left(\frac{t-3}{t+2}\right)^{2t+1}$; d) $\lim\limits_{x\to\frac{\pi}{4}}(\operatorname{tg}x)^{\operatorname{tg}2x}$

Rj. Znamo da je $\boxed{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e}$ $\boxed{\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e}$

a) $$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{n}=\left|\begin{array}{ll}\text{uvodimo smjenu} & \\ \frac{n}{a}=x & n\to\infty\Rightarrow x\to\infty \\ n=ax & \end{array}\right|=$$

$$=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{ax}=\lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{x}\right]^{a}=$$

$$=\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{a}=e^{a}$$

b) $$\lim_{x\to 0}\sqrt[x]{1-2x}=\lim_{x\to 0}(1-2x)^{\frac{1}{x}}=\left|\begin{array}{l}\text{uvodimo smjenu} \\ -2x=t \\ x\to 0\Rightarrow t\to 0\end{array}\right|=$$

$$=\lim_{t\to 0}(1+t)^{-\frac{2}{t}}=\left[\lim_{t\to 0}(1+t)^{\frac{1}{t}}\right]^{-2}=e^{-2}$$

c) $$\lim_{t\to\infty}\left(\frac{t-3}{t+2}\right)^{2t+1}=\lim_{t\to\infty}\left(\frac{t+2-5}{t+2}\right)^{2t+1}=\lim_{t\to\infty}\left(1+\frac{-5}{t+2}\right)^{2t+1}$$

$$=\left|\begin{array}{ll}\text{uvodimo smjenu} & \\ -\frac{5}{t+2}=x & t\to\infty\Rightarrow x\to 0 \\ -5=x(t+2) & 2t+4=-\frac{10}{x} \\ -\frac{5}{x}=t+2 & 2t+1=-\frac{10}{x}-3\end{array}\right|=\lim_{x\to\infty}(1+x)^{-\frac{10}{x}-3}=$$

$$= \lim_{x\to 0}\left[\left((1+x)^{\frac{1}{x}}\right)^{-10}\cdot(1+x)^{-3}\right] =$$

$$= \left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right]^{-10}\cdot \lim_{x\to 0}(1+x)^{-3} = e^{-10}\cdot 1 = \frac{1}{e^{10}}$$

d) $\displaystyle \lim_{x\to\frac{\pi}{4}}(\operatorname{tg} x)^{\operatorname{tg} 2x} = \Bigg|$ uvedimo smjenu $\operatorname{tg} x = 1+t$, $x\to\frac{\pi}{4} \Rightarrow t\to 0$

$\operatorname{tg} x = 1+t$

$$\operatorname{tg} 2x = \frac{\sin 2x}{\cos 2x} = \frac{2\sin x\cos x}{\cos^2 x-\sin^2 x}\quad \begin{matrix} /:\cos^2 x \\ /:\cos^2 x\end{matrix}$$

$$\operatorname{tg} 2x = \frac{2\operatorname{tg} x}{1-\operatorname{tg}^2 x} = \frac{2(1+t)}{1-(1+t)^2} = \frac{2(1+t)}{1-(1+2t+t^2)} = -\frac{2(t+1)}{t(t+2)}\Bigg| =$$

$$= \lim_{t\to 0}(1+t)^{-\frac{2(t+1)}{t(t+2)}} = \lim_{t\to 0}\left[(1+t)^{\frac{1}{t}}\right]^{-\frac{2(t+1)}{t+2}} = e^{-1}$$

zato što $\displaystyle \lim_{t\to 0}\frac{-2(t+1)}{t+2} = \frac{-2}{2} = -1$

(#) Izračunati $\lim_{x\to-\infty}\left(\frac{x+1}{3x+2}\right)^x$

Rj. Znamo da $\lim u^v = \lim u^{\lim v}$.

$$\lim_{x\to-\infty}\left(\frac{x+1}{3x+2}\right)^x = \lim_{x\to-\infty}\left(\frac{1+\frac{1}{x}}{3+\frac{2}{x}}\right)^x = \left(\frac{1}{3}\right)^{\lim_{x\to-\infty} x} = \left(\frac{1}{3}\right)^{-\infty}$$

$$= 3^{\infty} = \infty$$

(#) Izračunati $\lim\limits_{x\to 0}\left(\frac{1}{\sin 2x \sin x}-\frac{1}{2\sin^2 x}\right)$.

Rj.

$$\frac{1}{\sin 2x \sin x}-\frac{1}{2\sin^2 x}=\frac{2\sin x-\sin 2x}{2\sin 2x \sin^2 x}=$$

$$=\frac{2\sin x-2\sin x\cos x}{2\cdot 2\sin x\cos x(1-\cos^2 x)}=\frac{\cancel{2\sin x}\,\cancel{(1-\cos x)}}{2\cdot\cancel{2}\,\cancel{\sin x}\cos x\,\cancel{(1-\cos x)}(1+\cos x)}$$

$$=\frac{1}{2}\cdot\frac{1}{\cos x(1+\cos x)}$$

$$\lim_{x\to 0}\left(\frac{1}{\sin 2x\sin x}-\frac{1}{2\sin^2 x}\right)=\frac{1}{2}\lim_{x\to 0}\frac{1}{\cos x(1+\cos x)}=$$

$$=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$$

Jednostrani limesi

Ako je $x<a$ i $x\to a$, tada po dogovoru pišemo $x\to a-0$; analogno, ako je $x>a$ i $x\to a$, pišemo to ovako $x\to a+0$.
Brojeve $f(a-0)=\lim\limits_{x\to a-0} f(x)$ i $f(a+0)=\lim\limits_{x\to a+0} f(x)$

nazivamo lijevi limes f-je $f(x)$ u tački a i desni limes f-je $f(x)$ u tački a (ako ti brojevi postoje).
Koriste se i sledeće dvije oznake

$$f(a+)=\lim_{x\to a+} f(x) \quad \text{i} \quad f(a-)=\lim_{x\to a-} f(x)$$

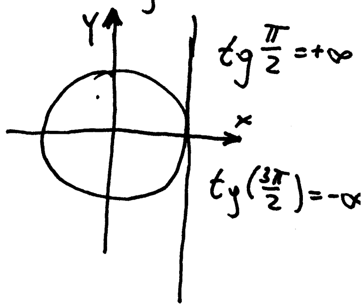
Za postojanje limesa f-je $f(x)$ kada $x\to a$ potrebno je i dovoljno da vrijedi jednakost $f(a-0)=f(a+0)$.

(1.) Izračunati desni i lijevi limes f-je $f(x)=\operatorname{arc\,tg}\frac{1}{x}$

Rj. $f(+0)=\lim\limits_{x\to +0} \operatorname{arc\,tg}\frac{1}{x}=\frac{\pi}{2}$

$f(-0)=\lim\limits_{x\to -0} \operatorname{arc\,tg}\frac{1}{x}=-\frac{\pi}{2}$

limes f-je $f(x)$ kad $x\to 0$ u ovom slučaju ne postoji



(2.) Izračunati jednostrane limese

a) $\lim\limits_{x\to -0}\frac{1}{1+e^{\frac{1}{x}}}=\frac{1}{1+e^{-\infty}}=\frac{1}{1+\frac{1}{e^{\infty}}}=1$ b) $\lim \frac{1}{1+e^{\frac{1}{x}}}$ Rj. 0

c) $\lim\limits_{x\to 2+0}\frac{x}{x-2}=\frac{2+0}{2+0-2}=\frac{2+0}{+0}=+\infty$ d) $\lim\limits_{x\to 2-0}\frac{x}{x-2}$ Rj. $-\infty$

e) $\lim\limits_{x\to -0}\frac{|\sin x|}{x}=\lim\limits_{x\to -0}\frac{-\sin x}{x}=-1$ f) $\lim\limits_{x\to +0}\frac{|\sin x|}{x}$ Rj. 1

g) $\lim\limits_{x\to 1-0}\frac{x-1}{|x-1|}=\lim\limits_{x\to 1-0}\frac{(x-1)}{-(x-1)}=\lim\limits_{x\to 1-0}(-1)=-1$ h) $\lim\limits_{x\to 1+0}\frac{x-1}{|x-1|}$ Rj. 1

i) $\lim\limits_{x\to -\infty}\frac{x}{\sqrt{x^2+1}}=\lim\limits_{x\to -\infty}\frac{x}{\sqrt{x^2}}=\lim\limits_{x\to -\infty}\frac{x}{|x|}=\lim\limits_{x\to -\infty}-\frac{x}{x}=-1$ j) $\lim\limits_{x\to +\infty}\frac{x}{\sqrt{x^2+1}}$ Rj. 1

Granična vrijednost funkcije

Neka je data realna funkcija $f : R \to R$;

Pojam granične vrijednosti funkcije

Za neku funkciju $y = f(x)$ kažemo da ima graničnu vrijednost A u tački a ako je

$$\begin{aligned} |f(x) - A| &< \varepsilon \\ |x - a| &< \delta(\varepsilon) \end{aligned} \qquad \text{i pišemo: } \lim_{x \to a} f(x) = A$$

Nreka su $f(x)$ i $g(x)$ i $\lim_{x \to a} f(x) = A$ i $\lim_{x \to a} g(x) = B$ tada važi:

$$\lim_{x \to a}(f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = A \pm B$$

$$\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = A \cdot B$$

$$\lim_{x \to a}(c \cdot f(x)) = c \cdot \lim_{x \to a} f(x) = cA$$

$$\lim_{x \to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \frac{A}{B}$$

$$\lim_{n \to +\infty}(a_n)^k = \left(\lim_{n \to +\infty} a_n\right)^k = a^k \quad k \neq \pm\infty$$

$$\lim_{n \to +\infty}\sqrt[k]{a_n} = \sqrt[k]{\lim_{n \to +\infty} a_n} = \sqrt[k]{a} \quad k \neq \pm\infty$$

$$\lim_{n \to +\infty} k^{a_n} = k^{\lim\limits_{n \to +\infty} a_n} = k^a \quad k \neq \pm\infty$$

$$\lim_{x \to 0}(1 + x)^{\frac{1}{x}} = e \qquad \lim_{x \to 0}\frac{\sin x}{x} = 1 \qquad \lim_{x \to 0}\frac{a^x - 1}{x} = \ln a$$

ZADACI

1. Naći:

$$\lim_{x \to +\infty}\frac{\sqrt{x}}{\sqrt{x + \sqrt{x + \sqrt{x}}}}$$

Rješenje

$$\lim_{x \to +\infty}\frac{\sqrt{x}}{\sqrt{x + \sqrt{x + \sqrt{x}}}} = \lim_{x \to +\infty}\frac{\frac{\sqrt{x}}{\sqrt{x}}}{\frac{\sqrt{x + \sqrt{x + \sqrt{x}}}}{\sqrt{x}}} = \lim_{x \to +\infty}\frac{1}{\sqrt{\frac{x + \sqrt{x + \sqrt{x}}}{x}}} = \lim_{x \to +\infty}\frac{1}{\sqrt{1 + \frac{\sqrt{x + \sqrt{x}}}{x}}} =$$

$$\lim_{x \to +\infty}\frac{1}{\sqrt{1 + \sqrt{\frac{x + \sqrt{x}}{x^2}}}} = \lim_{x \to +\infty}\frac{1}{\sqrt{1 + \sqrt{\frac{1}{x} + \frac{\sqrt{x}}{x^2}}}} = \lim_{x \to +\infty}\frac{1}{\sqrt{1 + \sqrt{\frac{1}{x} + \sqrt{\frac{1}{x^3}}}}} = \underline{\underline{1}}$$

2. Naći:

$$\lim_{x\to a}\frac{x^2-(a+1)x+a}{x^3-a^3}$$

Rješenje

$$\lim_{x\to a}\frac{x^2-(a+1)x+a}{x^3-a^3}=\lim_{x\to a}\frac{x^2-ax-x+a}{(x-a)(x^2+ax+a^2)}=\lim_{x\to a}\frac{x(x-a)-(x-a)}{(x-a)(x^2+ax+a^2)}=$$

$$\lim_{x\to a}\frac{x-1}{x^2+ax+a^2}=\underline{\underline{\frac{a-1}{3a^2}}}$$

3. Naći:

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$$

Rješenje

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}=\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}\cdot\frac{\sqrt[3]{x^2}+\sqrt[3]{x}+1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}\cdot\frac{\sqrt[4]{x}+1}{\sqrt[4]{x}+1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}=\lim_{x\to 1}\frac{x-1}{x-1}\cdot\frac{(\sqrt[4]{x}+1)(\sqrt{x}+1)}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}=$$

$$\lim_{x\to 1}\frac{(\sqrt[4]{x}+1)(\sqrt{x}+1)}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}=\underline{\underline{\frac{4}{3}}}$$

4. Naći:

$$\lim_{x\to 3}\frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}$$

Rješenje

$$\lim_{x\to 3}\frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}=$$

$$\lim_{x\to 3}\frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{(x-3)(x-1)}\cdot\frac{\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6}}{\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6}}=$$

$$\lim_{x\to 3}\frac{-4x+12}{(x-3)(x-1)\left(\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6}\right)}=\lim_{x\to 3}\frac{-4}{(x-1)\left(\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6}\right)}=$$

$$\frac{-4}{2(3+3)}=\underline{\underline{-\frac{1}{3}}}$$

5. Naći

$$\lim_{x\to\frac{\pi}{3}}\frac{1-2\cos x}{\pi-3x}$$

Rješenje

$$\lim_{x\to\frac{\pi}{3}}\frac{1-2\cos x}{\pi-3x}=\left(\frac{0}{0}\right)=\lim_{x\to\frac{\pi}{3}}\frac{2\left(\frac{1}{2}-\cos x\right)}{\pi-3x}=\lim_{x\to\frac{\pi}{3}}\frac{2\left(\cos\frac{\pi}{3}-\cos x\right)}{\pi-3x}=$$

$$\lim_{x\to\frac{\pi}{3}}\frac{2\left(-2\sin\frac{\frac{\pi}{3}+x}{2}\sin\frac{\frac{\pi}{3}-x}{2}\right)}{\pi-3x}=\lim_{x\to\frac{\pi}{3}}\frac{-4\sin\frac{\pi+3x}{6}\sin\frac{\pi-3x}{6}}{6\cdot\frac{\pi-3x}{6}}=\frac{-4\sin\frac{\pi+\pi}{6}}{6}=-\frac{\sqrt{3}}{3}$$

6. Naći

$$\lim_{x\to 1}\frac{\cos\frac{\pi x}{2}}{1-\sqrt{x}}$$

Rješenje

$$\lim_{x\to 1}\frac{\cos\frac{\pi x}{2}}{1-\sqrt{x}}=\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{\sin\left(\frac{\pi}{2}-\frac{\pi x}{2}\right)}{1-\sqrt{x}}\cdot\frac{1+\sqrt{x}}{1+\sqrt{x}}=\lim_{x\to 1}\frac{\sin\left[\frac{\pi}{2}(1-x)\right](1+\sqrt{x})}{1-x}=$$

$$\lim_{x\to 1}\frac{\sin\left[\frac{\pi}{2}(1-x)\right](1+\sqrt{x})}{\frac{2}{\pi}\cdot\frac{\pi}{2}(1-x)}=\frac{1\cdot 2}{\frac{2}{\pi}}=\pi$$

7. Naći:

$$\lim_{x\to+\infty}\left(\frac{1+x}{2+x}\right)^{\frac{\sqrt{x}-x\sqrt{x}}{1-\sqrt{x}}}$$

Rješenje

$$\lim_{x\to+\infty}\left(\frac{1+x}{2+x}\right)^{\frac{\sqrt{x}-x\sqrt{x}}{1-\sqrt{x}}}=\lim_{x\to+\infty}\left(\frac{2+x-1}{2+x}\right)^{\frac{\sqrt{x}(1-\sqrt{x})(1+\sqrt{x})}{1-\sqrt{x}}}=\lim_{x\to+\infty}\left(1-\frac{1}{2+x}\right)^{\sqrt{x}(1+\sqrt{x})}=$$

$$\lim_{x\to+\infty}\left\{\left[\left(1-\frac{1}{2+x}\right)^{2+x}\right]^{\frac{1}{2+x}}\right\}^{x+\sqrt{x}}=\left(e^{-1}\right)^{\lim\limits_{x\to+\infty}\frac{x+\sqrt{x}}{2+x}}=\left(e^{-1}\right)^{\lim\limits_{x\to+\infty}\frac{1+\sqrt{\frac{1}{x}}}{1+\frac{2}{x}}}=e^{-1}$$

8. Naći

$$\lim_{x\to 0}\left(\frac{1+tgx}{1+\sin x}\right)^{\frac{1}{\sin x}}$$

Rješenje

$$\lim_{x\to 0}\left(\frac{1+\mathrm{tgx}}{1+\sin x}\right)^{\frac{1}{\sin x}}=\lim_{x\to 0}\left(\frac{1+\sin x-\sin x+\mathrm{tgx}}{1+\sin x}\right)^{\frac{1}{\sin x}}=\lim_{x\to 0}\left(1-\frac{\sin x-\mathrm{tgx}}{1+\sin x}\right)^{\frac{1}{\sin x}}=$$

$$\lim_{x\to 0}\left[\left(1-\frac{\sin x-\mathrm{tgx}}{1+\sin x}\right)^{\frac{1+\sin x}{\sin x-\mathrm{tgx}}}\right]^{\frac{\sin x-\mathrm{tgx}}{1+\sin x}\cdot\frac{1}{\sin x}}=e^{-\lim\limits_{x\to 0}\frac{\sin x-\mathrm{tgx}}{1+\sin x}\cdot\frac{1}{\sin x}}=e^{-\lim\limits_{x\to 0}\frac{1-\frac{1}{\cos x}}{1+\sin x}}=e^{0}=1$$

******moguće su štamparske greške******

225. $\lim\limits_{x\to\infty}\dfrac{x^2-3x+1}{2x^2+x}=\frac{\infty}{\infty}=\lim\limits_{x\to\infty}\dfrac{(x^2-3x+1)/x^2}{(2x^2+x)/x^2}=\lim\limits_{x\to\infty}\dfrac{1-\dfrac{3}{x}+\dfrac{1}{x^2}}{2+\dfrac{1}{x}}=\dfrac{1}{2}.$

226. $\lim\limits_{x\to\infty}\dfrac{3x^4-5x^2+7x}{x^4-x^3+5}=\frac{\infty}{\infty}=\lim\limits_{x\to\infty}\dfrac{(3x^4-5x^2+7x)/x^4}{(x^4-x^3+5)/x^4}=\lim\limits_{x\to\infty}\dfrac{3-\dfrac{5}{x^2}+\dfrac{7}{x^3}}{1-\dfrac{1}{x}+\dfrac{5}{x^4}}=3.$

227. $\lim\limits_{x\to\infty}\dfrac{5x^2-x+3}{3x^3+2x-4}=\frac{\infty}{\infty}=\lim\limits_{x\to\infty}\dfrac{(5x^2-x+3)/x^3}{(3x^3+2x-4)/x^3}=\lim\limits_{x\to\infty}\dfrac{\dfrac{5}{x}-\dfrac{1}{x^2}+\dfrac{3}{x^3}}{3+\dfrac{2}{x^2}-\dfrac{4}{x^3}}=\dfrac{0}{3}=0.$

228. $\lim\limits_{x\to\infty}\dfrac{6x^4-2x^3+x^2}{2x^3+x^2-3}=\frac{\infty}{\infty}=\lim\limits_{x\to\infty}\dfrac{(6x^4-2x^3+x^2)/x^4}{(2x^3+x^2-3)/x^4}=\lim\limits_{x\to\infty}\dfrac{6-\dfrac{2}{x}+\dfrac{1}{x^2}}{\dfrac{2}{x}+\dfrac{1}{x^2}-\dfrac{3}{x^4}}=\dfrac{6}{0}=\infty.$

229. $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2-2x}}{x+1}=\frac{\infty}{\infty}=\lim\limits_{x\to\infty}\dfrac{(\sqrt{x^2-2x})/x}{(x+1)/x}=\lim\limits_{x\to\infty}\dfrac{\sqrt{1-\dfrac{2}{x}}}{1+\dfrac{1}{x}}=1.$

230. $\lim\limits_{x\to\infty}\dfrac{x+\sqrt{x^2-x}}{2x+3}=\frac{\infty}{\infty}=\lim\limits_{x\to\infty}\dfrac{(x+\sqrt{x^2-x})/x}{(2x+3)/x}=\lim\limits_{x\to\infty}\dfrac{1+\sqrt{1-\dfrac{1}{x}}}{2+\dfrac{3}{x}}=\dfrac{1+1}{2}=1.$

231. $\lim\limits_{x\to\infty}\dfrac{\sqrt{x+3}+\sqrt[4]{x^2-3x+1}}{2\sqrt{x-4}+\sqrt[4]{x^2-5}}=\frac{\infty}{\infty}=\lim\limits_{x\to\infty}\dfrac{[\sqrt{x+3}+\sqrt[4]{x^2-3x+1}]/\sqrt{x}}{[2\sqrt{x-4}+\sqrt[4]{x^2-5}]/\sqrt{x}}=$

$=\lim\limits_{x\to\infty}\dfrac{\sqrt{1+\dfrac{3}{x}}+\sqrt[4]{1-\dfrac{3}{x}+\dfrac{1}{x^2}}}{2\sqrt{1-\dfrac{4}{x}}+\sqrt[4]{1-\dfrac{5}{x^2}}}=\dfrac{1+1}{2+1}=\dfrac{2}{3}.$

232. $\lim\limits_{x\to\infty}\dfrac{\sqrt[5]{x^5-2x^3+4}+(3x-4)}{\sqrt[3]{x^3+x^2-4}+\sqrt{x^2-1}}=\frac{\infty}{\infty}=\lim\limits_{x\to\infty}\dfrac{[\sqrt[5]{x^5-2x^3+4}+(3x-4)]/x}{[\sqrt[3]{x^3+x^2-4}+\sqrt{x^2-1}]/x}=$

$=\lim\limits_{x\to\infty}\dfrac{\sqrt[5]{1-\dfrac{2}{x^2}+\dfrac{4}{x^5}}+3-\dfrac{4}{x}}{\sqrt[3]{1+\dfrac{1}{x}-\dfrac{4}{x^3}}+\sqrt{1-\dfrac{1}{x^2}}}=\dfrac{1+3}{1+1}=\dfrac{4}{2}=2.$

233. $\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2-2x}}{x+1}=\frac{\infty}{-\infty}=|\,x\to(-x)\,|=\lim\limits_{x\to\infty}\dfrac{\sqrt{(-x)^2-2(-x)}}{-x+1}=\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+2x}}{-x+1}=$

$$=\lim_{x\to\infty}\frac{(\sqrt{x^2+2x})/x}{(-x+1)/x}=\lim_{x\to\infty}\frac{\sqrt{1+\frac{2}{x}}}{-1+\frac{1}{x}}=-1.$$

234. $\lim\limits_{x\to-\infty}\dfrac{x-\sqrt{x^2+3x}}{2x+1}=\frac{-\infty}{-\infty}=|\,x\to(-x)\,|=\lim\limits_{x\to\infty}\dfrac{(-x)-\sqrt{(-x)^2+3(-x)}}{2(-x)+1}=$

$$=\lim_{x\to\infty}\frac{-x-\sqrt{x^2-3x}}{-2x+1}\quad=\lim_{x\to\infty}\frac{[-x-\sqrt{x^2-3x}]/x}{[-2x+1]/x}=\lim_{x\to\infty}\frac{-1-\sqrt{1-\frac{3}{x}}}{-2+\frac{1}{x}}=\frac{-1-1}{-2}=1.$$

➢ ZADACI ZA VJEŽBU

235. $\lim\limits_{x\to\infty}\dfrac{2x^3-x^2+1}{x^3+2x^2-4}$.

236. $\lim\limits_{x\to\infty}\dfrac{4x^2-x+10}{x^3+x^2-1}$.

237. $\lim\limits_{x\to\infty}\dfrac{2x-1+\sqrt{x^2-x}}{3x+\sqrt{x^2+7}}$.

238. $\lim\limits_{x\to-\infty}\dfrac{2x-1+\sqrt{x^2-x}}{3x+\sqrt{x^2+7}}$.

239. $\lim\limits_{x\to\infty}\dfrac{\sqrt[3]{x^3-2x}+4x-1}{2\sqrt{x^2+3x}+x}$.

240. $\lim\limits_{x\to-\infty}\dfrac{\sqrt[3]{x^3-2x}+4x-1}{2\sqrt{x^2+3x}+x}$.

241. $\lim\limits_{x\to\infty}\dfrac{4\cdot\sqrt{x^2-5x+1}+\sqrt[3]{x^3+x^2-7}}{3x+4+\sqrt{x^2-2x+5}}$.

242. $\lim\limits_{x\to-\infty}\dfrac{4\cdot\sqrt{x^2-5x+1}+\sqrt[3]{x^3+x^2-7}}{3x+4+\sqrt{x^2-2x+5}}$.

➢ RJEŠENJA

$R235.$ 2. $R236.$ 0. $R237.$ $\frac{3}{4}$. $R238.$ $\frac{1}{2}$. $R239.$ $\frac{5}{3}$. $R240.$ -5.
$R241.$ $\frac{5}{4}$. $R242.$ $-\frac{3}{2}$.

➢ **4.2 NEODREĐENI OBLIK** $\infty-\infty$

U ovoj točki ćemo računati limese funkcija kod kojih se nakon uvrštavanja $x=\infty$ pojavljuje neodređeni oblik $\infty-\infty$. U tom slučaju je potrebno danu funkciju transformirati raznim "trikovima" (racionaliziranje, faktoriziranje, itd.) na oblik $\frac{\infty}{\infty}$, te nastaviti u smislu prelaza sa beskonačno velikih na konačne i proizvoljno male veličine (dijeljenje brojnika i nazivnika sa najvećom potencijom), što je objašnjeno u prethodnom poglavlju.

➢ RJEŠENI PRIMJERI

U slijedećim zadacima izračunati limese funkcija.

243. $\lim\limits_{x\to\infty}(\sqrt{x}-\sqrt{x-3})=\lim\limits_{x\to\infty}(\sqrt{x}-\sqrt{x-3})\dfrac{\sqrt{x}+\sqrt{x-3}}{\sqrt{x}+\sqrt{x-3}}=\lim\limits_{x\to\infty}\dfrac{x-x+3}{\sqrt{x}+\sqrt{x-3}}=$

$$=\lim_{x\to\infty}\frac{3}{\sqrt{x}+\sqrt{x-3}}=0.$$

244. $\lim\limits_{x\to\infty}(x-\sqrt{x^2-3x+4})=\lim\limits_{x\to\infty}(x-\sqrt{x^2-3x+4})\dfrac{x+\sqrt{x^2-3x+4}}{x+\sqrt{x^2-3x+4}}=$

$$=\lim_{x\to\infty}\frac{x^2-x^2+3x-4}{x+\sqrt{x^2-3x+4}}=\frac{\infty}{\infty}=\lim_{x\to\infty}\frac{(3x-4)/x}{[x+\sqrt{x^2-3x+4}]/x}=\lim_{x\to\infty}\frac{3-\frac{4}{x}}{1+\sqrt{1-\frac{3}{x}+\frac{4}{x^2}}}=\frac{3}{2}.$$

245. $\lim\limits_{x\to\infty}(\sqrt{x^2-4}-\sqrt{x^2-2x+5})=\lim\limits_{x\to\infty}(\sqrt{x^2-4}-\sqrt{x^2-2x+5})\dfrac{\sqrt{x^2-4}+\sqrt{x^2-2x+5}}{\sqrt{x^2-4}+\sqrt{x^2-2x+5}}=$

$$=\lim_{x\to\infty}\frac{x^2-4-x^2+2x-5}{\sqrt{x^2-4}+\sqrt{x^2-2x+5}}=\tfrac{\infty}{\infty}=\lim_{x\to\infty}\frac{(2x-9)/x}{[\sqrt{x^2-4}+\sqrt{x^2-2x+5}]/x}=$$

$$=\lim_{x\to\infty}\frac{2-\frac{9}{x}}{\sqrt{1-\frac{4}{x^2}}+\sqrt{1-\frac{2}{x}+\frac{5}{x^2}}}=\frac{2}{1+1}=1.$$

246. $\lim\limits_{x\to\infty}\dfrac{1}{\sqrt{x}(\sqrt{x-3}-\sqrt{x+4})}=\lim\limits_{x\to\infty}\dfrac{1}{\sqrt{x}(\sqrt{x-3}-\sqrt{x+4})}\dfrac{\sqrt{x-3}+\sqrt{x+4}}{\sqrt{x-3}+\sqrt{x+4}}=$

$$=\lim_{x\to\infty}\frac{\sqrt{x-3}+\sqrt{x+4}}{\sqrt{x}(x-3-x-4)}=-\frac{1}{7}\lim_{x\to\infty}\frac{\sqrt{x-3}+\sqrt{x+4}}{\sqrt{x}}=\tfrac{\infty}{\infty}=$$

$$=-\frac{1}{7}\lim_{x\to\infty}\frac{[\sqrt{x-3}+\sqrt{x+4}]/\sqrt{x}}{\sqrt{x}/\sqrt{x}}=-\frac{1}{7}\lim_{x\to\infty}[\sqrt{1-\frac{3}{x}}+\sqrt{1+\frac{4}{x}}]=-\frac{2}{7}.$$

247. $\lim\limits_{x\to-\infty}(x+\sqrt{x^2-x+2})=\ |x\to(-x)|\ =\lim\limits_{-x\to-\infty}[(-x)+\sqrt{(-x)^2-(-x)+2}]\ =$

$$=\lim_{x\to\infty}[-x+\sqrt{x^2+x+2}]=\lim_{x\to\infty}[-x+\sqrt{x^2+x+2}]\frac{x+\sqrt{x^2+x+2}}{x+\sqrt{x^2+x+2}}=$$

$$=\lim_{x\to\infty}\frac{-x^2+x^2+x+2}{x+\sqrt{x^2+x+2}}=\tfrac{\infty}{\infty}=\lim_{x\to\infty}\frac{(x+2)/x}{[x+\sqrt{x^2+x+2}]/x}=$$

$$=\lim_{x\to\infty}\frac{1+\frac{2}{x}}{1+\sqrt{1+\frac{1}{x}+\frac{2}{x^2}}}=\frac{1}{1+1}=\frac{1}{2}.$$

➢ ZADACI ZA VJEŽBU

248. $\lim\limits_{x\to-\infty}(x+\sqrt{x^2-3x+1})$.

249. $\lim\limits_{x\to\infty}(\sqrt{x^2-3x+1}-x)$.

250. $\lim\limits_{x\to\infty}(\sqrt{x^2+x+1}-\sqrt{x^2+5x})$.

251. $\lim\limits_{x\to\infty}(x-\sqrt[3]{x^3+2x^2-1})$.

252. $\lim\limits_{x\to\infty}(\sqrt[4]{x^4+x^3-2}-\sqrt[4]{x^4-x^2+3x})$.

➢ RJEŠENJA

$R248.\ \ \frac{3}{2}.\quad R249.\ \ -\frac{3}{2}.\quad R250.\ \ -2.\quad R251.\ \ -\frac{2}{3}.\quad R252.\ \ \frac{1}{4}.$

➢ 4.3 NEODREĐENI OBLIK 1^{∞}

U ovoj točki računamo limese funkcija oblika $y=f(x)^{g(x)}$ kod kojih nakon uvrštavanja $x=\infty$ dobivamo oblik 1^{∞}. Osim svojstava limesa, nabrojanih na početku ovog poglavlja, koristit ćemo važan identitet:

$$\boxed{\lim_{x\to\infty}(1+\frac{1}{x})^x=e.}$$

Nadalje, treba primijeniti određene «trikove» pomoću kojih se dani oblik $y=f(x)^{g(x)}$ transformira na eksponencijalni oblik $e^{\frac{\infty}{\infty}}$, pa potom u eksponentu primijeniti rješavanje oblika $\frac{\infty}{\infty}$ s početka ovog poglavlja.

➢ RJEŠENI PRIMJERI

U slijedećim zadacima izračunati limese funkcija.

253. $\lim\limits_{x\to\infty}\left(\frac{x+3}{x}\right)^x=1^{\infty}=\lim\limits_{x\to\infty}\left(1+\frac{3}{x}\right)^x=\lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^x=\lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{\frac{x}{3}3}=e^3.$

254. $\lim\limits_{x\to\infty}\left(\frac{x+a}{x}\right)^x=1^{\infty}=\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^x=\lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x}{a}}\right)^x=\lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x}{a}}\right)^{\frac{x}{a}a}=$

$$=\left(\lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{a}}\right)^{\frac{x}{a}}\right)^a=e^a.$$

255. $\lim\limits_{x\to\infty}\left(\dfrac{x}{x+4}\right)^{x}=1^{\infty}=\lim\limits_{x\to\infty}\dfrac{1}{\left(\dfrac{x+4}{x}\right)^{x}}=\dfrac{1}{\lim\limits_{x\to\infty}\left(\dfrac{x+4}{x}\right)^{x}}=\dfrac{1}{e^{4}}=e^{-4}.$

256. $\lim\limits_{x\to\infty}\left(\dfrac{x}{x+a}\right)^{x}=1^{\infty}=\lim\limits_{x\to\infty}\dfrac{1}{\left(\dfrac{x+a}{x}\right)^{x}}=\dfrac{1}{\lim\limits_{x\to\infty}\left(\dfrac{x+a}{x}\right)^{x}}=\dfrac{1}{e^{a}}=e^{-a}.$

257. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+5}{x^2}\right)^{x^2}=1^{\infty}=\lim\limits_{x\to\infty}\left(1+\dfrac{5}{x^2}\right)^{x^2}=\lim\limits_{x\to\infty}\left(1+\dfrac{1}{\frac{x^2}{5}}\right)^{x^2}=\lim\limits_{x\to\infty}\left(1+\dfrac{1}{\frac{x^2}{5}}\right)^{\frac{x^2}{5}5}=e^{5}.$

258. $\lim\limits_{x\to\infty}\left(\dfrac{x-3}{x+1}\right)^{x}=1^{\infty}=\lim\limits_{x\to\infty}\left(1+\dfrac{x-3}{x+1}-1\right)^{x}=\lim\limits_{x\to\infty}\left(1+\dfrac{-4}{x+1}\right)^{x}=$

$=\lim\limits_{x\to\infty}\left(1+\dfrac{1}{\frac{x+1}{-4}}\right)^{\frac{x+1}{-4}(\frac{-4x}{x+1})}=e^{\lim\limits_{x\to\infty}\frac{-4x}{x+1}}=e^{-4}.$

259. $\lim\limits_{x\to\infty}\left(\dfrac{x-a}{x-b}\right)^{x}=1^{\infty}=\lim\limits_{x\to\infty}\left(1+\dfrac{x-a}{x-b}-1\right)^{x}=\lim\limits_{x\to\infty}\left(1+\dfrac{b-a}{x-b}\right)^{x}=$

$=\lim\limits_{x\to\infty}\left(1+\dfrac{1}{\frac{x-b}{b-a}}\right)^{\frac{x-b}{b-a}(\frac{(b-a)x}{x-b})}=e^{(b-a)\lim\limits_{x\to\infty}\frac{x}{x-b}}=e^{b-a}.$

➢ ZADACI ZA VJEŽBU

260. $\lim\limits_{x\to\infty}\left(\dfrac{\sqrt{x}+2}{\sqrt{x}-1}\right)^{2\sqrt{x}}.$

261. $\lim\limits_{x\to\infty}\left(\dfrac{x^3+x}{x^3+4}\right)^{3x^2}.$

262. $\lim\limits_{x\to\infty}\left(\dfrac{x^2-x}{x^2+3x-1}\right)^{x}.$

263. $\lim\limits_{x\to\infty}\left(\dfrac{x-2\sqrt{x}+3}{x+\sqrt{x}-1}\right)^{\sqrt{x}}.$

LIMES NIZA

Def: $\lim\limits_{n\to\infty} a_n = a$ ako $(\forall \varepsilon>0)(\exists n_0=n_0(\varepsilon)\in \mathbb{N})$ tako da je $n\geq n_0 \Rightarrow |a_n-a|<\varepsilon$

razlika vrlo mala blizu nule

ε-ima jako puno def. i dokaza teorema

najčešće ga koriste kada se hoće označi kao pozitivan broj, ali malo veći od 0. Znači ne nula, nego malo veći.

$a_n \approx a$

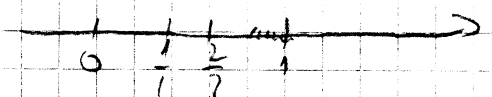
za velike indekse n - što je veći indeks bliži smo limesu.

Pr. $a_n = \frac{n}{n+1}$ $\quad \lim\limits_{n\to\infty} a_n = 1$

$a_3 = \frac{3}{4}$; $\left|\frac{3}{4}-1\right| = \frac{1}{4}$

$a_{10} = \frac{10}{11}$; $\left|\frac{10}{11}-1\right| = \frac{1}{11}$ (manji od $\frac{1}{4}$)

Što je veći indeks n član niza je sve bliži tom limesu.



Što smo bliže jedinici razlika između članova je manja.

Limesu se može približavati na 2 načina

kada raste sa lijeva na desno

kada opada sa desna na lijevo

konvergentan niz ne mora biti konvergentan, ali je svaki put bliže limesu a

- Kod limesa pokazuju kada n teži ∞, uvijek da rezultat bude

∞ samo postoji + ili −.

- Najveći stepen ima i najbrži rast u ∞

1. $\lim\limits_{n\to\infty} (n^2+9n-7) = \lim\limits_{n\to\infty} n^2 = +\infty$

2. $\lim\limits_{n\to\infty} (2n^2-n^3-5n+11) = \lim\limits_{n\to\infty} (-n^3) = -\infty$

3. $\lim\limits_{n\to\infty} \dfrac{4n^3+2n^2-1}{3n^3+7} = \lim\limits_{n\to\infty} \dfrac{4n^3}{3n^3} = \dfrac{4}{3}$

dijelimo s polinomom najvećeg stepena, odnosno s najvećim stepenom

4. $\lim\limits_{n\to\infty} \dfrac{2n+1}{n^2-5} = \lim\limits_{n\to\infty} \dfrac{2n}{n^2} = \dfrac{2}{n} = 0$, kada u brojniku imamo konstantu, a u nazivniku

$\dfrac{c}{\infty} = 0$ (c = constanta)

konstanta

Ukoliko dobijemo nešto u brojniku ili nazivniku, tada izračunati nemo prvo

5. $\lim\limits_{n\to\infty} \dfrac{1+2+3+\dots+n}{n^2}$ suma uzastopnih n

$= \lim\limits_{n\to\infty} \dfrac{\frac{n(n+1)}{2}}{n^2}$

$= \lim\limits_{n\to\infty} \dfrac{n(n+1)}{2n^2}$

$= \lim\limits_{n\to\infty} \dfrac{n+1}{2n}$

$= \lim\limits_{n\to\infty} \dfrac{n}{2n} = \dfrac{1}{2}$

za ispita

6. $\lim\limits_{n\to\infty} \dfrac{1^2+2^2+3^2+\dots+(3n+2)^2}{8n^3-(2n-1)^3} = S_2(n)$

$1^2+2^2+\dots+n^2 = \boxed{\dfrac{n(n+1)(2n+1)}{6}}$ — NAPAMET!!!

$1^2+2^2+\dots+(3n+2)^2 = \dfrac{(3n+2)(3n+2+1)\cdot[2\cdot(3n+2)+1]}{6}$

$\lim\limits_{n\to\infty} = \dfrac{\frac{(3n+2)(3n+3)(6n+5)}{6}}{8n^3-(8n^3-3\cdot 4n^2+3\cdot 2n-1)}$ u ovakvoj zadaći uzeti najveći

$= \lim\limits_{n\to\infty} \dfrac{\frac{3n\cdot 3n\cdot \cancel{6n}}{\cancel{6}}}{\cancel{8n^3}-\cancel{8n^3}+12n^2-6n+1} = \lim\limits_{n\to\infty} \dfrac{9n^3}{12n^2} = \lim\limits_{n\to\infty} \dfrac{3n}{4} = +\infty$

7) $\lim\limits_{n\to\infty} \dfrac{1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1)}{n^3}$

ako je sabiraka dva izraza, sum od tog, onda rastavljamo na 2 sume

$S(n) = 1\cdot 2+2\cdot 3+\cdots+n(n+1)$

$= \sum\limits_{k=1}^{m} k(k+1) =$ skraćeno napisano $= \sum\limits_{k=1} (k^2+k) =$

$= \sum\limits_{k=1}^{m} k^2 + \sum\limits_{k=1}^{m} k =$

$= S_2(m) + S_1(m) = \dfrac{n(n+1)(2n+1)}{6} + \dfrac{n(n+1)}{2}$

$= \dfrac{n(n+1)(2n+1)+3n(n+1)}{6}$

$= \dfrac{n(n+1)(2n+1+3)}{6} = \dfrac{n(n+1)(2n+4)}{6}$

$= \dfrac{\cancel{2}n(n+1)(n+2)}{\cancel{6}} = \dfrac{n(n+1)(n+2)}{3}$

$\lim\limits_{n\to\infty} \dfrac{\frac{n(n+1)(n+2)}{3}}{n^3} = \lim\limits_{n\to\infty} \dfrac{n\cdot n\cdot n}{3n^3} = \dfrac{1}{3}$

do ove sam znam

do ovdje - mislim odavde

Za vježbu:

a) $\lim\limits_{n\to\infty} \left[\dfrac{1+3+5+\cdots+(2n-1)}{n+1} - \dfrac{2n+1}{2}\right]$?

b) $\lim\limits_{n\to\infty} \dfrac{1^2+2^2+\cdots+(n-1)^2}{n^2}$

c) $\lim\limits_{n\to\infty} \dfrac{1\cdot 2^2+2\cdot 3^2+\cdots+n(n+1)^2}{1^2\cdot 2+2^2\cdot 3+\cdots+n^2(n+1)}$

Uputa: $\lim\limits_{n\to\infty} \dfrac{\sum\limits_{k=1}^{n} k(k+1)^2}{\sum\limits_{k=1}^{n} k^2(k+1)}$

za ispita

d)

$$\lim_{n\to\infty} \frac{5+9+13\ldots+(8n-3)}{2+5+8+\ldots+(6n+11)}$$

aritmetički niz
proučiti formulu za sumu aritmetičkog niza

Postupak racionalizacije

paziti koeficijent onda ovaj kvadrat se racionalizira ako ovo ne može

(8.) $\lim_{n\to\infty} \sqrt{n}\,(\underbrace{\sqrt{n+3}}_{+\infty}-\underbrace{\sqrt{n+1}}_{+\infty})$

- ne može se primijeniti postupak izlučivanja konstante zbog minusa ($tip: +\infty-\infty$), može ako je ispred + ili ako je sa $\sqrt{}$ samostalno zato je $+\infty$

$$=\lim_{n\to\infty} \sqrt{n}\,(\sqrt{n+3}-\sqrt{n+1})\cdot\frac{\sqrt{n+3}+\sqrt{n+1}}{\sqrt{n+3}+\sqrt{n+1}}$$

$$=\lim_{n\to\infty} \sqrt{n}\cdot\frac{(\sqrt{n+3})^2-(\sqrt{n+1})^2}{\sqrt{n+3}+\sqrt{n+1}}$$

$$=\lim_{n\to\infty} \frac{\sqrt{n}\cdot(n+3-n-1)}{\sqrt{n+3}+\sqrt{n+1}}$$

$$=\lim_{n\to\infty} \frac{2\sqrt{n}}{\sqrt{n}+\sqrt{n}}=\lim_{n\to\infty}\frac{2\sqrt{n}}{2\sqrt{n}}=1$$

odbacim se pa se mogu skratiti

(9) $\lim_{n\to\infty} (\underbrace{n}_{a}-\sqrt[3]{\underbrace{n^3-5n^2+4n+7}_{b}})$ $(+\infty,+\infty)$

$$=\left(a^3-b^3=(a-b)\cdot(a^2+ab+b^2)\right)$$

uradi po ovoj formuli

$$=\lim_{n\to\infty}\frac{(n-\sqrt[3]{n^3-5n^2+4n+7})\cdot\left[n^2+n\cdot\sqrt[3]{n^3}+\left(\sqrt[3]{n^3}\right)^2\right]}{n^2+n\cdot\sqrt[3]{\quad}+\left(\sqrt[3]{\quad}\right)^2}$$

$$\lim_{n\to\infty}\frac{n^3-\left(\sqrt[3]{n^3-5n^2+4n+7}\right)^3}{n^2+n\sqrt[3]{\quad}+\left(\sqrt[3]{\quad}\right)^2}=\lim_{n\to\infty}\frac{5n^2-4n-7}{n^2+n\sqrt[3]{\quad}+\left(\sqrt[3]{\quad}\right)^2}$$

ostavimo najveći član

$$=\lim_{n\to\infty}\frac{5n^2}{n^2+n\cdot\sqrt[3]{n^3}+\left(\sqrt[3]{n^3}\right)^2}=\lim_{n\to\infty}\frac{5n^2}{n^2+n^2+n^2}=\lim_{n\to\infty}\frac{5n^2}{3n^2}=\frac{5}{3}$$

Za vjezbu: a - b

a) $\lim\limits_{n \to \infty} \left(\sqrt{n+\sqrt{n}} - \sqrt{n-\sqrt{n}} \right)$

Iskoristi razliku kvadrata

b) $\lim\limits_{n \to \infty} \left(\sqrt[4]{n+1} - \sqrt[4]{n} \right)$ – 2 puta r. k.

c) $\lim\limits_{n \to \infty} n \cdot \left(1 - \sqrt[3]{1 - \frac{1}{n}} \right)$ sa ispita

d) $\lim\limits_{n \to \infty} \left(n - \sqrt[3]{n^3 - n^2 + 1} \right)$ razlika kubova

$$\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$$

10. $\lim\limits_{n \to \infty} \Big(\underbrace{\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)}}_{S(n)} \Big)$ svaki razlomak, kao razliku 2 razl.

$$S(n) = \frac{2-1}{1\cdot 2} + \frac{3-2}{2\cdot 3} + \frac{4-3}{3\cdot 4} + \dots + \frac{(n+1)-n}{n(n+1)}$$

$$= \frac{2}{1\cdot 2} - \frac{1}{1\cdot 2} + \frac{3}{2\cdot 3} - \frac{2}{2\cdot 3} + \frac{4}{3\cdot 4} - \frac{3}{3\cdot 4} + \dots + \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)}$$

$$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \dots + \cancel{\frac{1}{n}} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

$$\lim_{n \to \infty} S(n) = 1$$

11. $\lim\limits_{n \to \infty} \Big(\underbrace{\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \dots + \frac{1}{(3n-2)(3n+1)}}_{S(n)} \Big)$ razlika pomoću prirodnih brojeva za 3, a u zagradi.

Uputa: $S(n) = \frac{1}{3} \left[\frac{4-1}{1\cdot 4} + \frac{7-4}{4\cdot 7} + \dots + \frac{(3n+1)-(3n-2)}{(3n-2)(3n+1)} \right]$.

Za vježbu:

a) $\lim\limits_{n\to\infty}\left(1+\frac{1}{1+2}+\frac{1}{1+2+3}+\dots+\frac{1}{1+2+\dots+n}\right)$

b) $\lim\limits_{n\to\infty}\left(1-\frac{1}{3}\right)\cdot\left(1-\frac{1}{6}\right)\cdots\left(1-\frac{1}{\frac{n(n-1)}{2}}\right)$ [illegible] ... u pitanju [illegible] ... se [illegible] napraviti.

c) $\lim\limits_{n\to\infty}\frac{1}{\sqrt[3]{n}}\cdot\left(\frac{1}{1+\sqrt[3]{2}+\sqrt[3]{3}}+\frac{1}{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}}+\dots+\frac{1}{\sqrt[3]{(n-1)^2}+\sqrt[3]{n(n-1)}+\sqrt[3]{n^2}}\right)$

$\frac{1}{\sqrt[3]{\infty}} \to 0$, $\lim\limits_{n\to\infty} = 0$ racionalizacija

d) $\lim\limits_{n\to\infty}\left(\frac{1}{\ln 2\cdot\ln 4}+\frac{1}{\ln 4\cdot\ln 8}+\dots+\frac{1}{\ln 2^{n-1}\cdot\ln 2^{n}}\right)$

$\ln 4 = \ln 2^2$, $\ln 8 = \ln 2^3$

$$\lim_{n\to\infty} a^n = \begin{cases} 0, & -1<a<1 \\ 1, & a=1 \\ +\infty, & a>1 \end{cases}$$

[illegible]

$\lim\limits_{n\to\infty} a^n$ ne postoji ako je $a\le -1$

[illegible] ne pogledi [illegible]

12. $\lim\limits_{n\to\infty}\frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}}\Big/ :3^{n+1} = \lim\limits_{n\to\infty}\frac{\frac{(-2)^n}{3^{n+1}}+\frac{\cancel{3^n}}{3^{\cancel{n+1}}}}{\frac{(-2)^{n+1}}{3^{n+1}}+1}$

$= \lim\limits_{n\to\infty}\frac{\frac{(-2)^n}{3^n\cdot 3}+\frac{1}{3}}{\left(-\frac{2}{3}\right)^{n+1}+1} = \lim\limits_{n\to\infty}\frac{\cancel{\left(-\frac{2}{3}\right)^n}\cdot\frac{1}{3}+\frac{1}{3}}{\cancel{\left(-\frac{2}{3}\right)^{n+1}}+1} = \frac{\frac{1}{3}}{1} = \frac{1}{3}$

[illegible]

Za vježbu:

$\lim\limits_{n\to\infty}\frac{1+a+a^2+\dots+a^n}{1+b+b^2+\dots+b^n}$, ako je $a^2<b^2<1$

Teorema o lopovom i 2 policajca

Ako je niz $b_n \le a_n \le c_n$ i $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = a$

$\Rightarrow \lim_{n\to\infty} a_n = a$

Zadaci:

1. $\lim_{n\to\infty} \sqrt[n]{2^n+2} = 2$

mit koji teži ka 2

$$\underbrace{\sqrt[n]{2^n}}_{2} < \sqrt[n]{2^n+2} < \left(2+\frac{1}{n}\right)^n$$

očigledno

$\Leftrightarrow 2^n+2 < \left(2+\frac{1}{n}\right)^n$ nije prirodan broj, pošto su potencije nego je teže raspisati pa se skoncentrišemo na prva dva člana

$\Leftrightarrow \cancel{2^n}+2 < \cancel{2^n} + \cancel{n}\cdot 2^{n-1}\cdot\frac{1}{\cancel{n}} + \dots$

(2.) za vježbu jedan od lakših

$\lim_{n\to\infty} \frac{\cos(n^2+n)}{n+1}$

$(-1,1)$ nema neke uloge. Šta god da je pod cos ili sin ovo zanemarimo

$$\frac{-1}{n+1} < \frac{\cos(n^2+n)}{n+1} < \frac{1}{n+1}$$

$\to 0$

svaka suma se ne može svesti, ...

(3.) $\lim_{n\to\infty} a_n, \quad a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}$

IDEJA: uoči koji je najveći, a koji najmanji i sa njima sve razlomke zamijeniti

$$a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}$$

najveći (zbog recipročne vrijednosti), najmanji

$$\frac{1}{\sqrt{n^2+n}} + \dots + \frac{1}{\sqrt{n^2+n}} < a_n < \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \dots + \frac{1}{\sqrt{n^2+1}}$$

$$\frac{n}{\sqrt{n^2+n}} < a_n < \frac{n}{\sqrt{n^2+1}}$$

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2}} = \underline{1}$$

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2}} = 1$$

$$\lim_{n\to\infty} a_n = \underline{1}$$

5. $\lim_{n\to\infty} \frac{n}{2^n}$

binomna formula

$$2^n = (1+1)^n = 1^n + \binom{n}{1} 1^{n-1} \cdot 1 + \binom{n}{2} 1^{n-2} \cdot 1^2 + \dots + 1^n > \binom{n}{2} = \frac{n\cdot(n-1)}{2}$$

zbog recipročne vrijednosti i nejednakosti

$$\Rightarrow \frac{n}{2^n} < \frac{n}{\frac{n(n-1)}{2}} = \frac{2n}{n(n-1)} = \frac{2}{n-1}$$

$$0 < \frac{n}{2^n} < \frac{2}{n-1}$$

$\frac{2}{n-1} \to 0$ teži nuli

$$\Rightarrow \lim_{n\to\infty} \frac{n}{2^n} = 0$$

Za vježbu! Dokažite da je $\lim_{n\to\infty} \frac{n}{a^n} = 0$ ako je $a > 1$, pomoću [illegible]

po binomnoj formuli

$$a^n = (\underbrace{a-1}_{>0}+1)^n = \dots$$

$$> \binom{n}{2} \dots$$